Cookbook/Summary for PDE

PDE Class of 2022

This is the summary/cookbook/ for the class PDE taught by Alden Waters in the academic year 2021-2022

As this is a collective project, if you plan to make use of it make sure to contribute as well.

If what you want to add doesn't have a section yet feel free to add it.

It is probably best to have some theory/description and then some examples for each section/topic.

Contents

1 Important Terminology and Definitions, Classifications

Types of PDE's

Definition 1.1. A PDE is said to be **linear** if u and all its derivatives appear linearly in it, otherwise it is said to be **nonlinear**

Definition 1.2. A nonlinear PDE is said to be **quasilinear** if the derivatives of principal order (the highest order) occour only linearly with coefficients which may depend on derivatives of lower order.

Definition 1.3. A quasilinear PDE is said to be **semilinear** if the coefficients depend only on x (explicitly) otherwise they are **fully nonlinear**.

Definition 1.4. We call a linear second order partial differential equation:

- **eliptic** at x_0 if all eigenvalues of $A(x_0)$ have the same sign.
- **parabolic** at x_0 if one or more eigenvalues of $A(x_0)$ vanish.
- **hyperbolic** at x_0 if none of the eigenvalues of $A(x_0)$ vanish, and all but one have the same sign.
- **ultrahyperbolic** in all other cases.

Remark 1.5. This course is concerned primarly with linear 2d order PDE's with the classifications eliptic, parabolic and hyperbolic.

2 Stationary, Transport and Traveling waves

Stationary waves

Transport equation

Definition 2.1. The transport equation is a linear, homogeneous first-order partial differential equation given as follows:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

If we have an initial condition $u(t_0, x) = f(x)$ for all $x \in \mathbb{R}$, with $f \in C^1$ then we have a unique solution to (2.1)

Proposition 2.2. *(Proposition 2.1 in book) If $u(t, x)$ is a solution to the partial differential equation*

$$u_t + cu_x = 0$$

which is defined on all of $\mathbb{R}^2$, then

$$u(t, x) = v(x - ct)$$

where $v(\xi)$ is a C^1 function of the characteristic variable $\xi = x - ct$.

Characteristic curves

3 The Wave Equation

The wave equation

Definition 3.1. The wave equation is a linear, second order, homogeneous partial differential equation given by:

$$u_{tt} = c^2 u_{xx}$$

d'Alembert's formula

Theorem 3.2. *(Theorem 2.15 in book) The solution to the initial value problem*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), \quad -\infty < x < \infty,$$

is given by

$$u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz$$

Note: for the above Thm to be a classical solution we need $f \in C^2$ and $g \in C^1$.

Examples 3.3.

Example 3.4.

Method of seperation of variables

If we take a look at the wave equation again, and try to find seperable solutions, i.e. $u(t,x) = w(t)v(x)$, plugging this in we get:

$$w''(t)v(x) = c^2 w(t)v''(x)$$

If we divide by $v(x)w(t)$ assuming that it is not 0 we get:

$$\frac{w''(t)}{w(t)} = c^2 \frac{v''(x)}{v(x)} = \lambda$$

This is two seperate ODE's which we already know how to solve and give us the following possible seperable solutions depending on λ.

λ	$w(t)$	$v(x)$	$u(t,x) = w(t)v(x)$
$\lambda = -\omega^2 < 0$	$\cos\omega t, \sin\omega t$	$\cos\frac{\omega x}{c}, \sin\frac{\omega x}{c}$	$\cos\omega t\cos\frac{\omega x}{c}, \cos\omega t\sin\frac{\omega x}{c}, \sin\omega t\cos\frac{\omega x}{c}, \sin\omega t\sin\frac{\omega x}{c}$
$\lambda = 0$	$1, t$	$1, x$	$1, x, t, tx$
$\lambda = \omega^2 > 0$	$e^{-\omega t}, e^{\omega t}$	$e^{-\omega x/c}, e^{\omega x/c}$	$e^{-\omega(t+x/c)}, e^{\omega(t-x/c)}, e^{-\omega(t-x/c)}, e^{\omega(t+x/c)}$

Table 1: Seperable Solutions to the Wave Equation

Examples 3.5.

Example 3.6.

d'Alembert's solution on bounded intervals

Examples 3.7.

Example 3.8.

4 Eigen solutions & Fourier Series

Rescaling a Fourier series

Convergence of Fourier series

Remark 4.1. Note: According to the Prof, the heat equation will be on the final exam, but not on the midterm.

5 The Heat Equation

Definition 5.1. The Heat Equation is given by:

$$u_t = \nabla u = u_x x + u_y y$$

Remark 5.2. Note: The Heat equation is an example of a parabolic PDE, and is one of the main equations for this course

The heated ring problem

The fundemental solution

The forced Heat Equation and Duhamel's Principle

6 The Delta Function

The Diffusion and Heat Equation

Dirichlet boundary conditions

Neumann boundary conditions

6.1 Robin boundary conditions

Remark 6.1. The Prof **highly** reccommended reviewing the method of seperation of variables on disks and rectangles before the midterm.

The Planar Laplace equation

Definition 6.2. The two-dimensional **Laplace equation** is the second-order linear partial differential equation

$$u_{xx} + u_{yy} = 0$$

6.2 BDV rectangles

To solve Laplace's equation, we try the method of sepearation of variables $u(x,t) = v(x)w(y)$ and plug it into our equation, this gives:

$$v''(x)w(y) + w''(y)v(x) = 0$$

Assuming that $v(x)w(y) \neq 0$ we divide and get

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda$$

where since one is a function of y and one is a function of x we have that λ is a constant. This gives us two ODE's which when we solve give the following separable solutions

λ	$w(y)$	$v(x)$	$u(t,x) = w(t)v(x)$
$\lambda = -\omega^2 < 0$	$e^{-\omega y}, e^{\omega y}$	$\cos\omega x, \sin\omega x$	$e^{\omega y}\cos\omega x, e^{\omega y}\sin\omega x, e^{-\omega y}\cos\omega x, e^{-\omega y}\sin\omega x$
$\lambda = 0$	$1, y$	$1, x$	$1, x, y, xy$
$\lambda = \omega^2 > 0$	$\cos\omega y, \sin\omega y$	$e^{-\omega x}, e^{\omega x}$	$e^{\omega x}\cos\omega y, e^{\omega x}\sin\omega y, e^{-\omega x}\cos\omega y, e^{-\omega x}\sin\omega y$

Table 2: Seperable Solutions to Laplace's Equation

Example 6.3. 4.3.11. (a) Explain how to use lienar superposition to solve the boundary value problem

$$\nabla u = 0, \quad u(x,0) = f(x), \quad u(x,b) = g(x), \quad u(0,y) = h(y), \quad u(a,y) = k(y)$$

on the rectangle $R = 0 < x < a, 0 < y < b$ by splitting it into four seperate boundary value problems for which each of the solutions vanishes on three sides of the rectangle.

> The four separate boundary value problems are given by $\nabla u = 0$ and for each side separately one non-zero boundary condition. The total solution is the sum of the solutions to each seperate boundary value problem.

(b) Write down a series formula for the resulting solution.

> First we solve each individual BVP we start with the one given by $u(x,0) = f(x)$. This implies that $u(0,y) = 0, u(a,y) = 0, u(x,b) = 0$
>
> The initial conditions $u(0,y) = 0 = u(a,y)$ imply that $v(x) = \sin(\omega x)$ and $\omega = n\pi/a$, with $n \in \mathbb{Z}_{>0}$ (Note: negative n is just a constant multiple of the positive n and so they are not unique solutions).
>
> Since $\lambda = -\omega^2 = -\frac{n^2\pi^2}{a^2} < 0$ we have that $w(y) = c_1 e^{-\omega y} + c_2 e^{\omega y}$ as we have $u(x,b) = 0$, this implies that $w(b) = 0$ using this gives $c_1 = e^{\omega b}, c_2 = -e^{-\omega b}$, and simplifying slightly, $w(y) = \sinh(\omega(b-y))$. Thus the seperable solutions are given by:
>
> $$\sin(n\pi/ax)\sinh(n\pi(b-y)/a)$$

Finally $u(x,0) = f(x)$ implies that we have:

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x/a) \sinh(k\pi b/a)$$

which implies that we can calculate c_k from the fourier series for $f(x)$ as follows:

$$c_k = \frac{1}{\sinh(k\pi b/a)} \frac{2}{a} \int_0^a f(x) \sin(k\pi x/a)$$

And the solution $u(x,y)$ is given by

$$u(x,y) = \sum_{k=1}^{\infty} c_k \sin(k\pi x/a) \sinh(k\pi(b-a)/a)$$

with c_k as above.

Now we do the same for $u(0,y) = h(y)$ which gives us $u(x,0) = 0$, $u(x,b) = 0$, $u(a,y) = 0$.

The initial conditions $u(x,0) = 0 = u(x,b)$ implies that $w(y) = \sin(\omega y)$ and $\omega = n\pi/b$ with $n \in \mathbb{Z}_{>0}$. As we are in the case $\lambda > 0$ this implies that $v(x) = c_1 e^{-\omega x} + c_2 e^{\omega x}$, using $u(a,y) = 0$ implies that $c_1 = e^{\omega b}$ and $c_2 = e^{-\omega b}$

The Poisson Equation

Definition 6.4. The forced Laplace equation or **Poisson's equation** is given by:

$$-\nabla[u] = -u_{xx} - u_{yy} = f(x,y)$$

7 Green's Function

Green's Functions are defined as

Green's Functions for One-Dimensional Boundary Value Problems

For second-order linear ordinary differential equations

$$L[u] = p(x)\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + r(x)u(x) = f(x)$$

with a pair of homogeneous boundary conditions at the ends of the interval $[a, b]$, and p, q, r, f continous and $p(x) \neq 0$ for all $x \in [a, b]$. the properties of the Green's Function $G(x; \xi)$ are as follows:

Let Ω be the domain, and $L[u] = f$ with $u|_{\partial\Omega} = 0$. Then the Green's functions satisfy the following properties

- $L[G(x, \xi)] = \delta(x - \xi)$
- $G(x, \xi)$ is continuous in Ω
- $G(x, \xi)$ satisfies the homogeneous boundary conditions
- $\partial G/\partial x$ is piecewise C' with a single jump discontinuity of magnitude $1/p(\xi)$ at the impulse point $x = \xi$

Green's Functions for the Planar Poisson Equation

Green's Functions and Convulation

8 Fourier Transform

The maximum principle

Definition 8.1. Let Ω be a bounded open connected domain. A function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is said to be subharmonic if $-\Delta u \leqslant 0$ in Ω, superharmonic if $-\Delta u \geqslant 0$ in Ω

Notes about the final exam

The professor has said that there will be 4 questions on the final exam.

2. Heat equation with separation of variables 3. Green's function 4. A repeat from an old exam.